Math 217.002 F25

Quiz 28 – Solutions

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1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) An *inner product* on a vector space V is ...

   Solution: (Over $\mathbb{R}$.) A function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ such that for all $u, v, w \in V$ and all $a, b \in \mathbb{R}$:

   - **Bilinearity:** $\langle au+bv,\ w\rangle = a\langle u, w\rangle + b\langle v, w\rangle$ and $\langle u,\ av+bw\rangle = a\langle u, v\rangle + b\langle u, w\rangle$;
   - **Symmetry:** $\langle u, v\rangle = \langle v, u\rangle$;
   - **Positive–definiteness:** $\langle v, v\rangle \geq 0$ with equality iff $v = 0$.

   (b) An *inner product space* is ...

   Solution: A vector space V together with a specified inner product $\langle\cdot,\cdot\rangle$ on V. We denote it by $(V, \langle\cdot,\cdot\rangle)$.

2. Prove that if $\mathcal{U} = (\vec{u}_1, \ldots, \vec{u}_n)$ is an orthonormal basis of the inner product space V, then

$$\langle \vec{x}, \vec{y}\rangle \;=\; [\vec{x}]_{\mathcal{U}} \cdot [\vec{y}]_{\mathcal{U}}$$

for all $\vec{x}, \vec{y} \in V$.

Solution: Because $\mathcal{U}$ is orthonormal, every $x \in V$ expands as

$$x = \sum_{i=1}^{n} \langle x, u_i\rangle\, u_i,$$

so $[x]_{\mathcal{U}} = (\langle x, u_1\rangle, \ldots, \langle x, u_n\rangle)^\top$. Similarly, $y = \sum_{j=1}^{n} \langle y, u_j\rangle\, u_j$. Then

$$\langle x, y\rangle = \Big\langle \sum_i \langle x, u_i\rangle u_i,\ \sum_j \langle y, u_j\rangle u_j \Big\rangle = \sum_{i,j} \langle x, u_i\rangle\, \langle y, u_j\rangle\, \langle u_i, u_j\rangle.$$

Orthonormality gives $\langle u_i, u_j\rangle = \delta_{ij}$, hence

$$\langle x, y\rangle = \sum_{i=1}^{n} \langle x, u_i\rangle\, \langle y, u_i\rangle \;=\; [x]_{\mathcal{U}} \cdot [y]_{\mathcal{U}}.$$

*For full credit, please write out fully what you mean instead of using shorthand phrases.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

   Suppose $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ is a basis of the inner product space V, then

$$\langle \vec{x}, \vec{y} \rangle \;=\; [\vec{x}]_{\mathcal{B}} \cdot [\vec{y}]_{\mathcal{B}}.$$

Solution: **FALSE.** The formula holds precisely when $\mathcal{B}$ is orthonormal. In general,

$$\langle x, y \rangle = [x]_{\mathcal{B}}^{\top}\, G_{\mathcal{B}}\, [y]_{\mathcal{B}}, \quad \text{where } (G_{\mathcal{B}})_{ij} = \langle v_i, v_j \rangle$$

is the Gram matrix. If $G_{\mathcal{B}} \neq I$, the standard dot product of coordinate vectors does not equal the inner product.

Counterexample in $\mathbb{R}^2$: Take the standard inner product and the basis $\mathcal{B} = (e_1,\ e_1 + e_2)$, which is not orthonormal. Let $x = y = e_2$. Solve $e_2 = ae_1 + b(e_1 + e_2)$ to get $b = 1$, $a = -1$, so $[e_2]_{\mathcal{B}} = (-1, 1)^{\top}$. Then

$$[e_2]_{\mathcal{B}} \cdot [e_2]_{\mathcal{B}} = (-1)^2 + 1^2 = 2 \quad \text{but} \quad \langle e_2, e_2 \rangle = 1.$$

Thus the claimed equality fails.